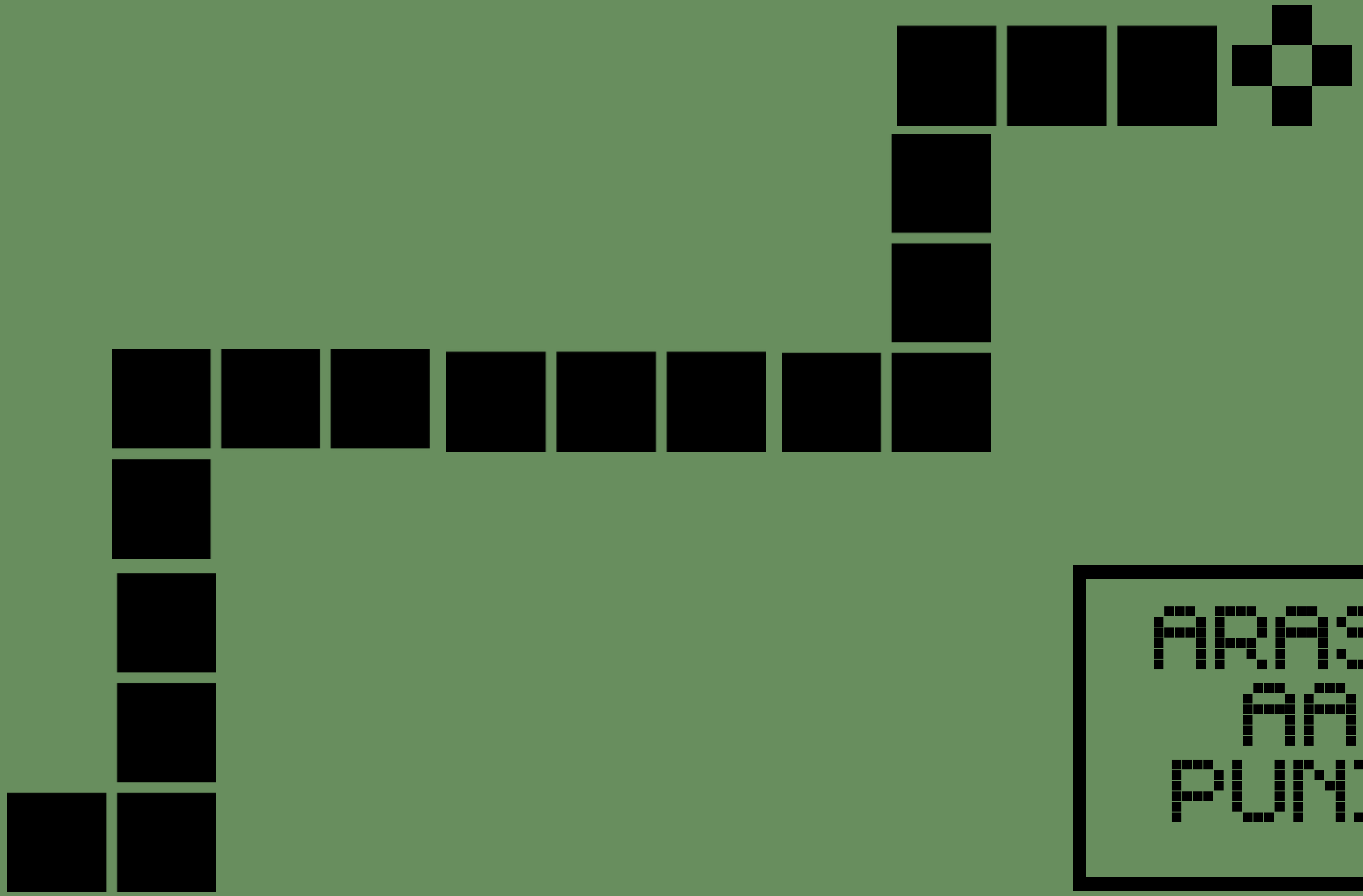
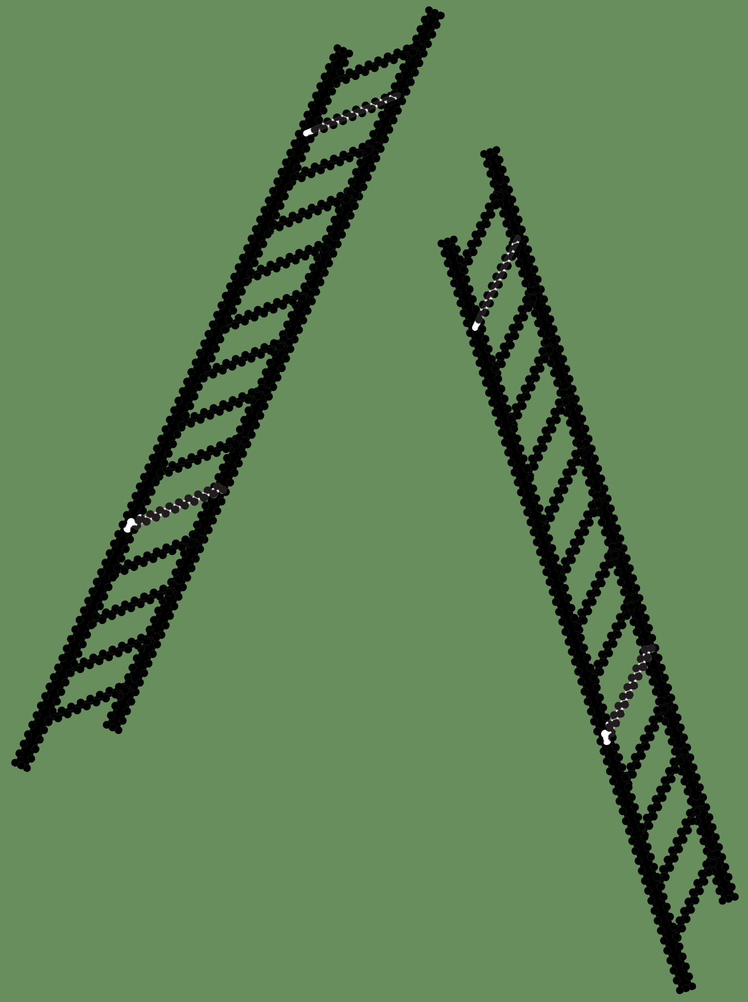
THRESHOLD RESETTING IN SNAKES AND LADDERS





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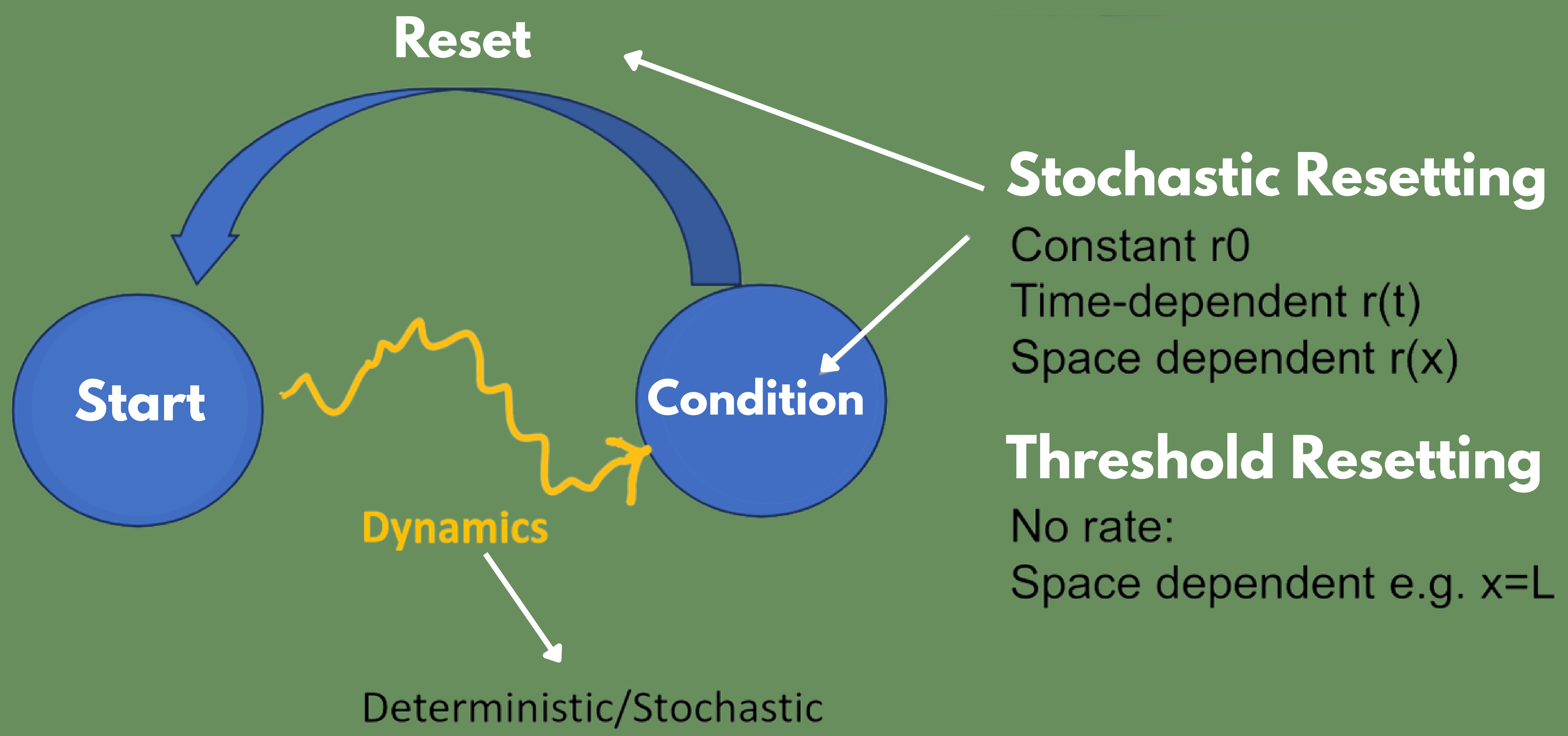
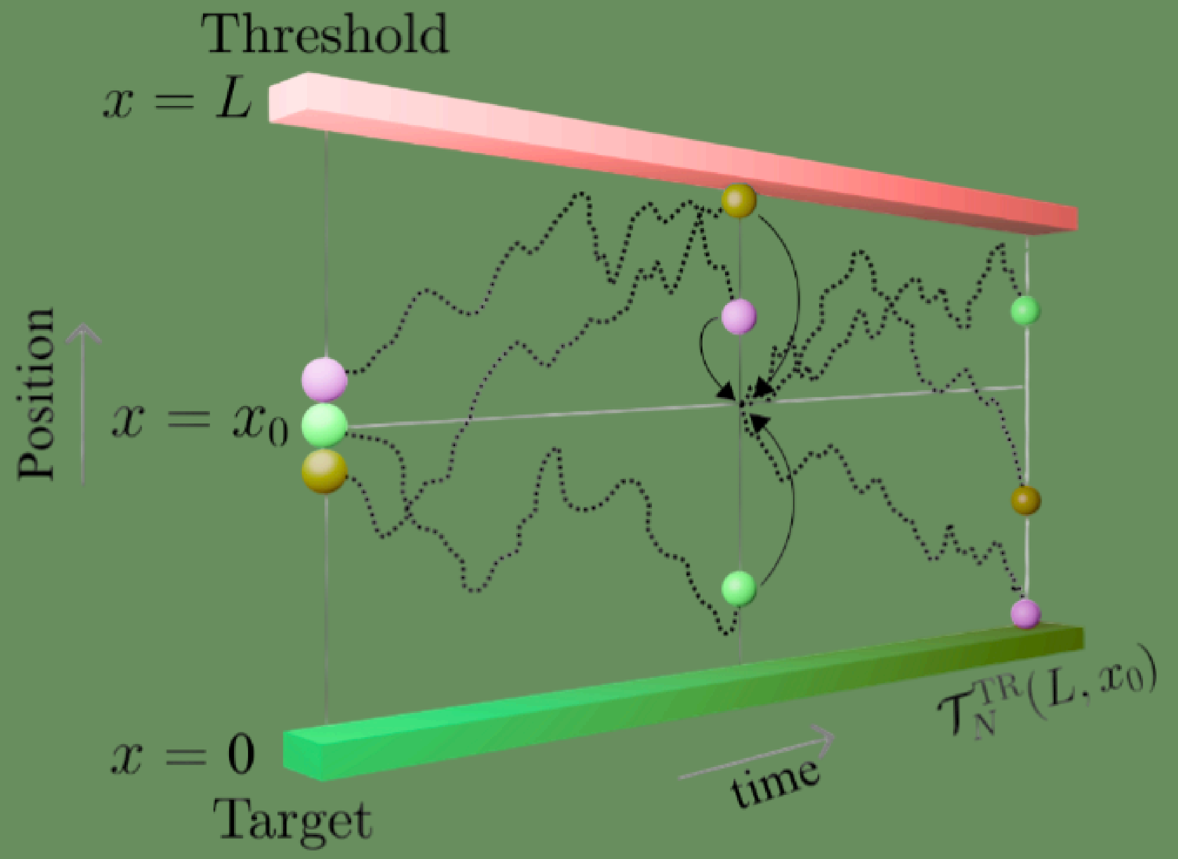
RESETTING

- Literally its name! sporadically reverting to their initial condition once a certain condition is met
- It's of 2 types : Stochastic and Threshold



THRESHOLD RESETTING



- It is a phenomenon in which resetting occurs when the system meets a certain threshold
- Consider the system on your left: N non-interacting searchers in [0,L] each governed with starting position x_0
- We can model this process with the following equation:

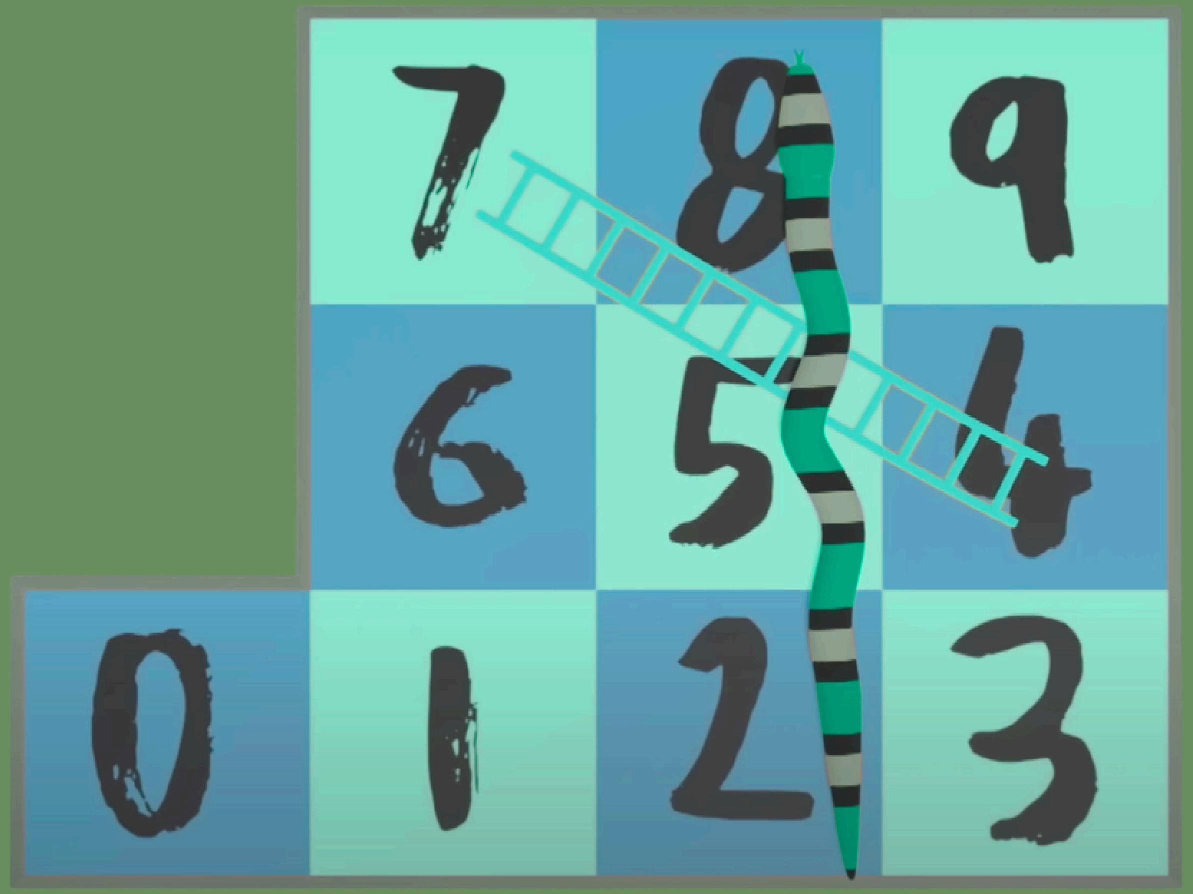
$$\mathcal{Q}_N^{TR}(L, x_0, t) = q_N(L, x_0, t) + \int_0^t dt' j_{L,N}(L, x_0, t') \mathcal{Q}_N^{TR}(L, x_0, t - t')$$

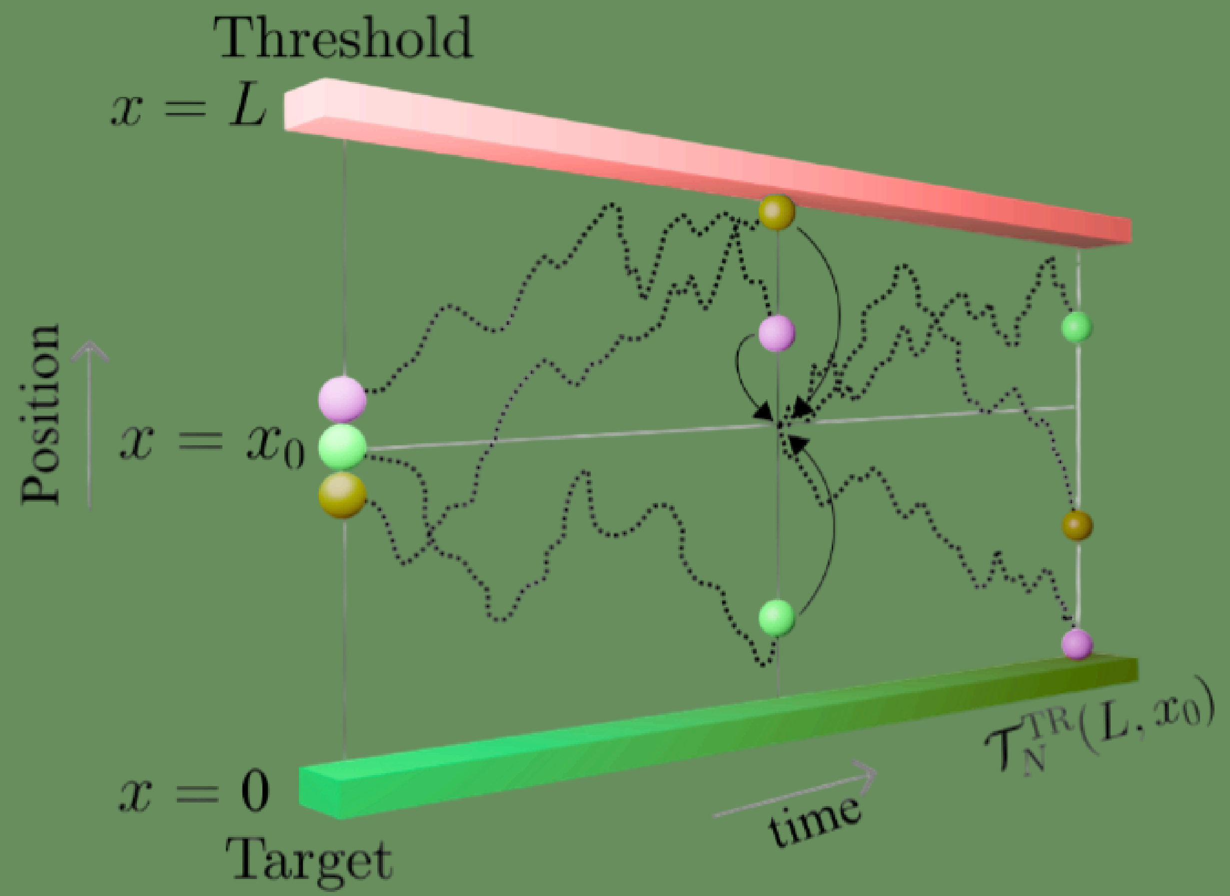
$\mathcal{Q}_N^{TR}(L, x_0, t)$	$q_N(L, x_0, t)$	$j_{L,N}(L, x_0, t')$	$\mathcal{Q}_N^{TR}(L, x_0, t - t')$
SURVIVAL PROBABILITY OF SEARCHERS	SURVIVAL PROBABILITY WITH NO RESETTING	PROBABILITY FLUX AT THRESHOLD L WITH t' THE TIME IT REACHES L	SURVIVAL PROBABILITY OF SEARCHERS

A. Biswas, S. N. Majumdar, and A. Pal, Phys. Rev. Lett. 135, 227101 (2025)

THE IDEA

The Idea came from the stochastic nature of the game and the fact that snakes essentially reset the position of the player to the start of the game and ladders act as a shorter path the snake can help you get to





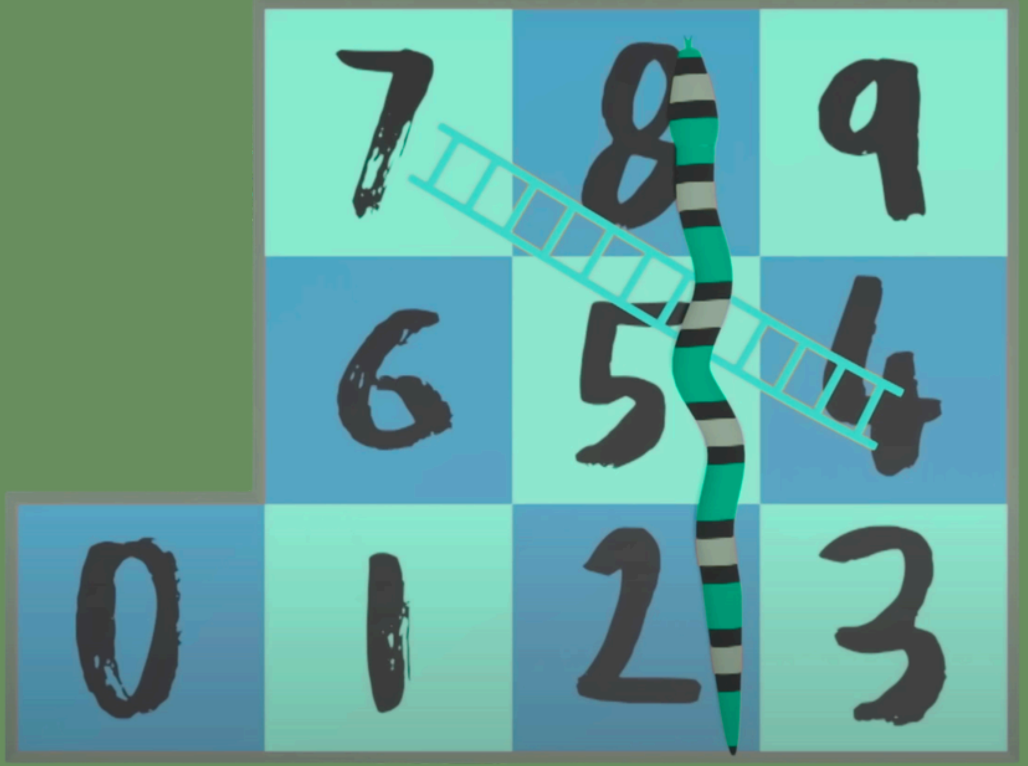
Numberphile, The Beautiful Math of Snakes and Ladders - Numberphile (2023), YouTube video, https://www.youtube.com/watch?v=nlm07asSU0c

ASSUMPTIONS I MADE

- For the snake to act as a reset, it needs to help reach the target early.
- The ladder should be thought of as a tunnel that takes the player closer to the target in fewer turns (i.e., with less movement).
- For the resetting to have any significant effect, the effect of diffusion (here, the randomness from the dice roll) needs to be reduced.
- We start with a one-ladder, one-snake game for simplicity.

MODELLING THE GAME

- Consider the snakes and ladder given on your right, we can model this game as a 1 searcher game which resets its position at 8



- Instead of time, out unit of measuring becomes the number of turns(T), N = 1(only 1 searcher) and i is the minimum number of turns required to reach the threshold(8)

- the meaning of Q^{TR} and q remain the same though the math required to find them would change. Similarly j_{reset} would represent the probability of the searcher hitting 8 at T' instead of density

Numberphile, The Beautiful Math of Snakes and Ladders - Numberphile (2023), YouTube video, https://www.youtube.com/watch?v=nIm07asSU0c

MODELLING THE GAME

$$Q^{\mathrm{TR}}(T) = q(T) + \sum_{T'=i}^{T} j_{\mathrm{reset}}(T')\, Q^{\mathrm{TR}}(T - T')$$

- We use First Passage Time (MFPT)[mean time taken to reach absorbing state, here 1000] and optimize it the get the optimal threshold limit.

- To find it say, j(T) is the probability it reaches Target(1000) at turn T.
- Therefore, the Mean passage time would be $\mathbb{E}[T] = \langle T \rangle = \sum_{T=0}^{\infty} Q^{\mathrm{TR}}(T)$

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MODELLING THE GAME

$$Q^{\mathrm{TR}}(T) = \sum_{T'=T+1}^{\infty} j(T') \longrightarrow \sum_{T=0}^{\infty} Q^{\mathrm{TR}}(T) = \sum_{T=0}^{\infty} \sum_{T'=T+1}^{\infty} j(T')$$

$$= \sum_{T'=1}^{\infty} j(T') \sum_{t=0}^{T'-1} 1 = \sum_{T'=1}^{\infty} T' \, j(T') = \sum_{T=1}^{\infty} T \, j(T) = \mathbb{E}[T]$$

- As we have a discrete system, our maths to find MFPT becomes easier. We introduce a transition Matrix M:

$$M = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

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THE MATHS BEHIND THE GAME

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 5 & 6 & 7 \\ \hline 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 2/6 \\ 2 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 2/6 \\ 3 & 0 & 0 & 1/6 & 0 & 1/6 & 1/6 & 2/6 \\ 5 & 0 & 0 & 1/6 & 0 & 2/6 & 1/6 & 1/6 \\ 6 & 0 & 0 & 1/6 & 0 & 0 & 3/6 & 1/6 \\ 7 & 0 & 0 & 1/6 & 0 & 0 & 0 & 4/6 \end{array}$$

- The image you see at your side is Q, 4 and 8 are not their as they are the ladder start and snake head respectively and 9 is the target. Now for the next bit, I need to define some terms:

$$P_s(T) = \Pr(\text{ball at state } s \text{ at turn } T)$$

$$R_s = \Pr(s \to 1000)$$

- **Therefore:** $j(T) = \sum_{s=0}^{999} P_s(T-1)\, R_s$

Numberphile, The Beautiful Math of Snakes and Ladders - Numberphile (2023), YouTube video, https://www.youtube.com/watch?v=nIm07asSU0c

THE MATHS BEHIND THE GAME

- Define $\mathbf{p}(T) = \begin{pmatrix} P_0(T) \\ P_1(T) \\ \vdots \\ P_{999}(T) \end{pmatrix}$

Now: $j(T) = \sum_{s=0}^{999} P_s(T-1)\, R_s$

Therefore: $j(T) = \mathbf{R}^\top \mathbf{p}(T-1)$

AS THE PROCESS IS STOCHASTIC AND THE STATE OF THE SEARCHER ONLY DEPENDS ON ITS PREVIOUS STATE, WE CAN MODEL THIS AS A A MARKOVIAN PROCESS

- Therefore we can say that $M_{ij} = \Pr(s_{T+1} = j \mid s_T = i)$

$$\mathbf{p}(T+1) = M\,\mathbf{p}(T)$$

Numberphile, The Beautiful Math of Snakes and Ladders - Numberphile (2023), YouTube video, https://www.youtube.com/watch?v=nlm07asSU0c

THE MATHS BEHIND THE GAME

- Now as: $\mathbf{p}(T+1) = Q\,\mathbf{p}(T)$
- We can write: $\mathbf{p}(T-1) = \mathbf{Q}^{T-1}\mathbf{p}(0)$
- And as: $j(T) = \mathbf{R}^\top\mathbf{p}(T-1)$
- We can write this as: $j(T) = \mathbf{R}^\top\,\mathbf{Q}^{T-1}\mathbf{p}(0)$

THEREFORE : $$Q^{\mathrm{TR}}(T) = \sum_{s=T+1}^{\infty} \mathbf{R}^\top\,\mathbf{Q}^{\,s-1}\mathbf{p}(0)$$

Numberphile, The Beautiful Math of Snakes and Ladders - Numberphile (2023), YouTube video, https://www.youtube.com/watch?v=nlm07asSU0c

THE MATRIX MATHEMATICS

$$\mathbf{R}^\top = 1^\top(I - \mathbf{Q}) \longrightarrow Q^{\mathrm{TR}}(T) = 1^\top(I - \mathbf{Q})\mathbf{p}(0)\mathbf{Q}^{\,T}\sum_{s=0}^{\infty}\mathbf{Q}^{\,k}$$

$$(\mathbf{I} - \mathbf{Q})(\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \ldots + \mathbf{Q}^n) = \mathbf{I} - \mathbf{Q}^{n+1}$$

$$(\mathbf{I} - \mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q})(\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \ldots + \mathbf{Q}^n) = (\mathbf{I} - \mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q}^{n+1})$$

$$\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \ldots + \mathbf{Q}^n = (\mathbf{I} - \mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q}^{n+1})$$

$$\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \ldots + = (\mathbf{I} - \mathbf{Q})^{-1}$$

THEREFORE : $Q^{\mathrm{TR}}(T) = \mathbf{1}^\top\mathbf{Q}^{\,T}\mathbf{p}(0)$

Hochman, M. (2014). Chutes and ladders. University of Chicago REU.

THE MATRIX MATHEMATICS

$$Q^{\mathrm{TR}}(T) = \mathbf{1}^\top \mathbf{Q}^{\,T} \mathbf{p}(0) \longrightarrow \langle T \rangle = \sum_{T=0}^{\infty} \mathbf{1}^\top \mathbf{Q}^T \mathbf{p}(0)$$

THEREFORE : $\boxed{\langle T \rangle = \mathbf{1}^\top (I - \mathbf{Q})^{-1} \mathbf{p}(0)}$

Hochman, M. (2014). Chutes and ladders. University of Chicago REU.

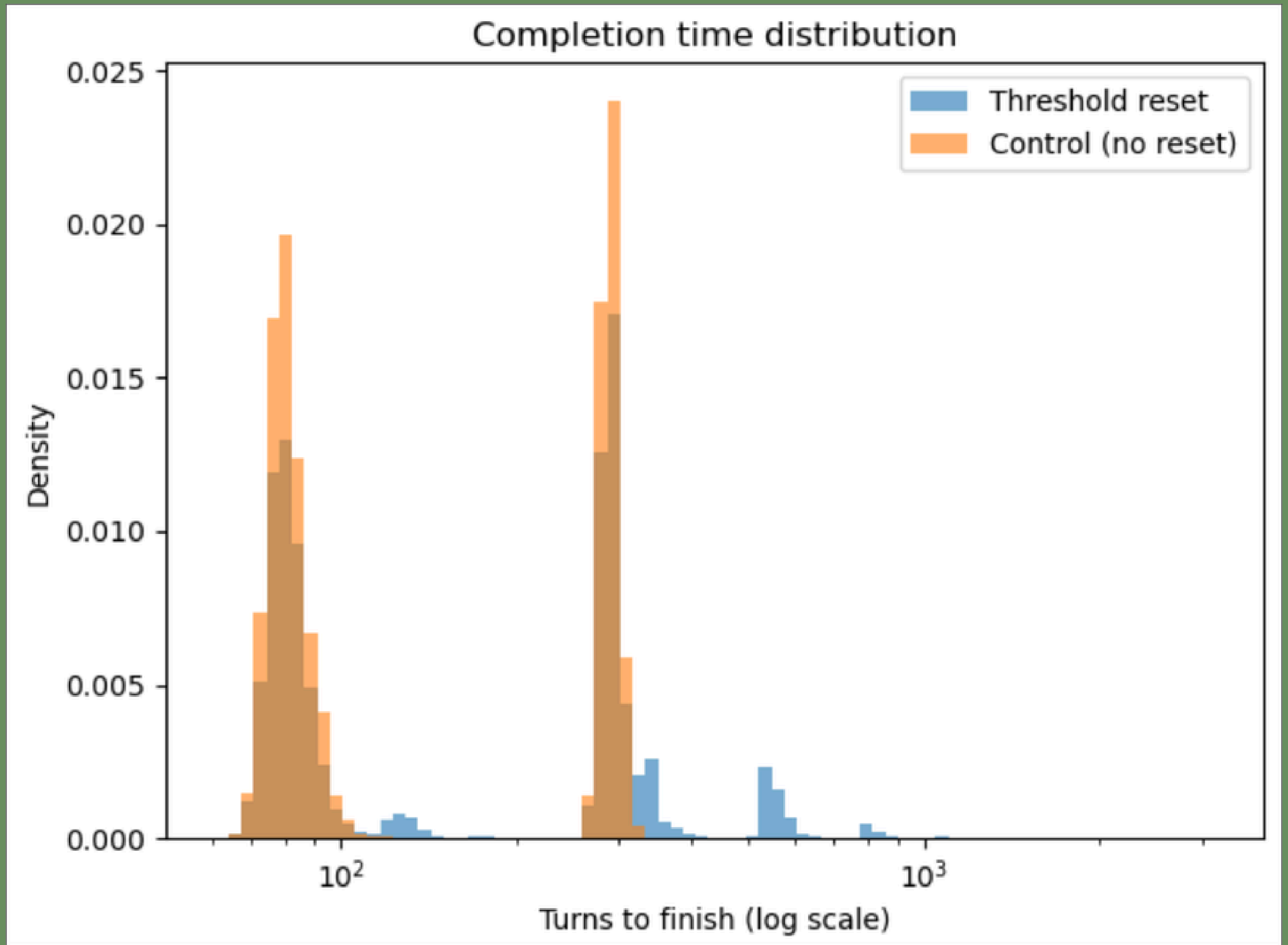
MOVING TO SIMULATIONS

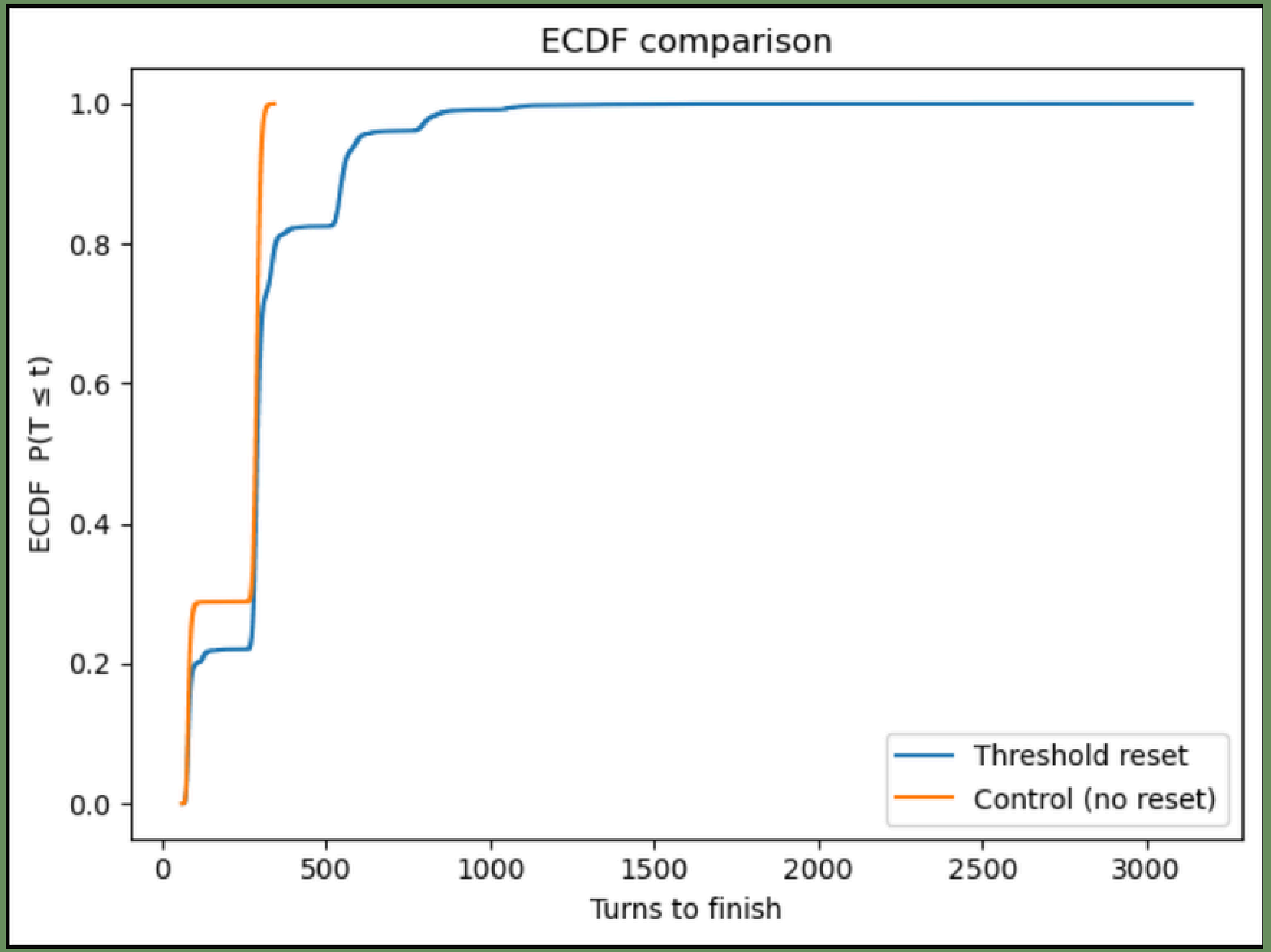
We ran a simulation of N runs for a game and tried to find the following:

- ECDF (Empirical Cumulative Distribution Function) is essentially the fraction of games completed before turn T
- Δ Mean Turns represents the difference in the mean number of turns required to complete the game after and before resetting

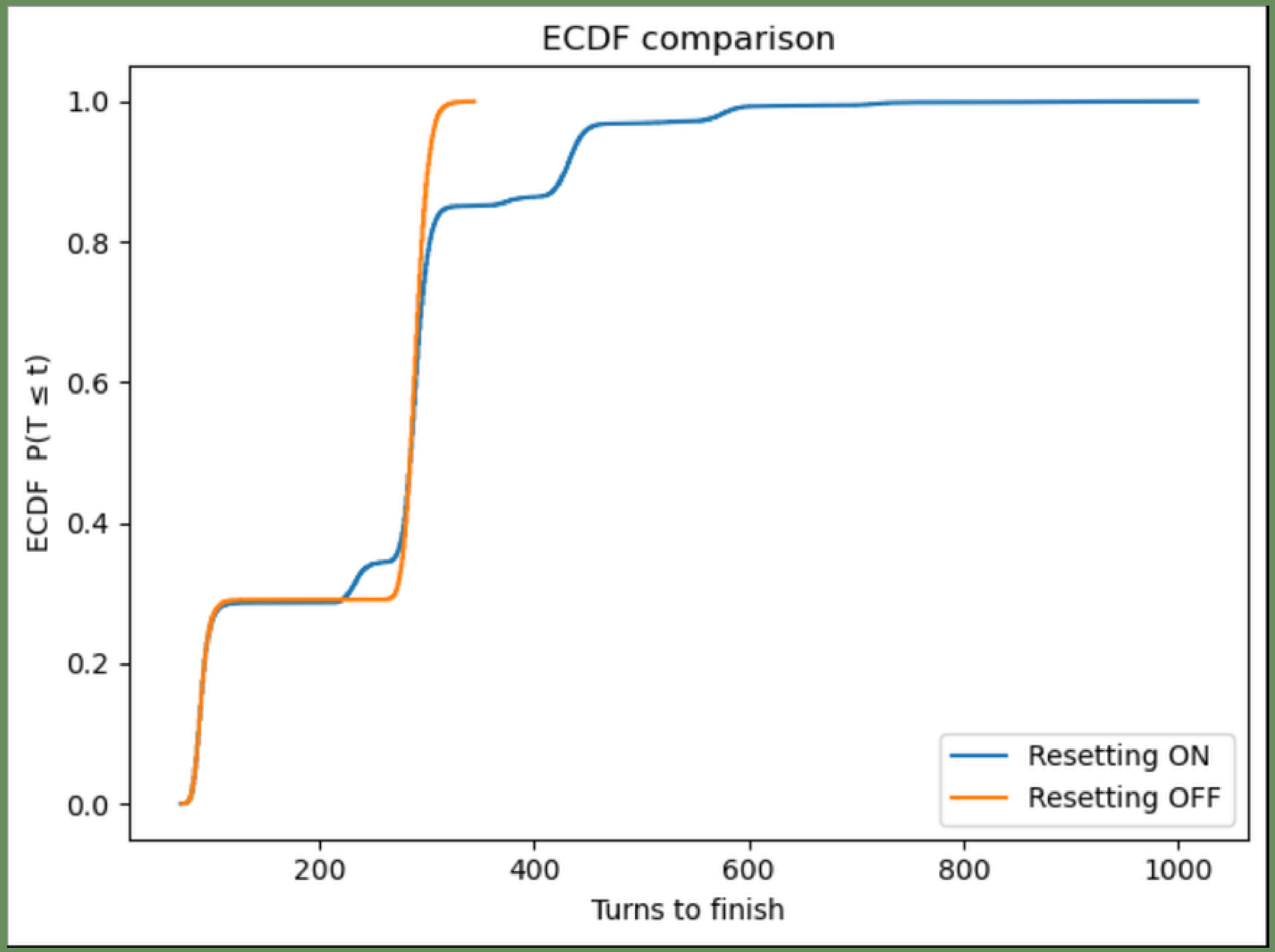
FIRST TRIAL N = 30,000

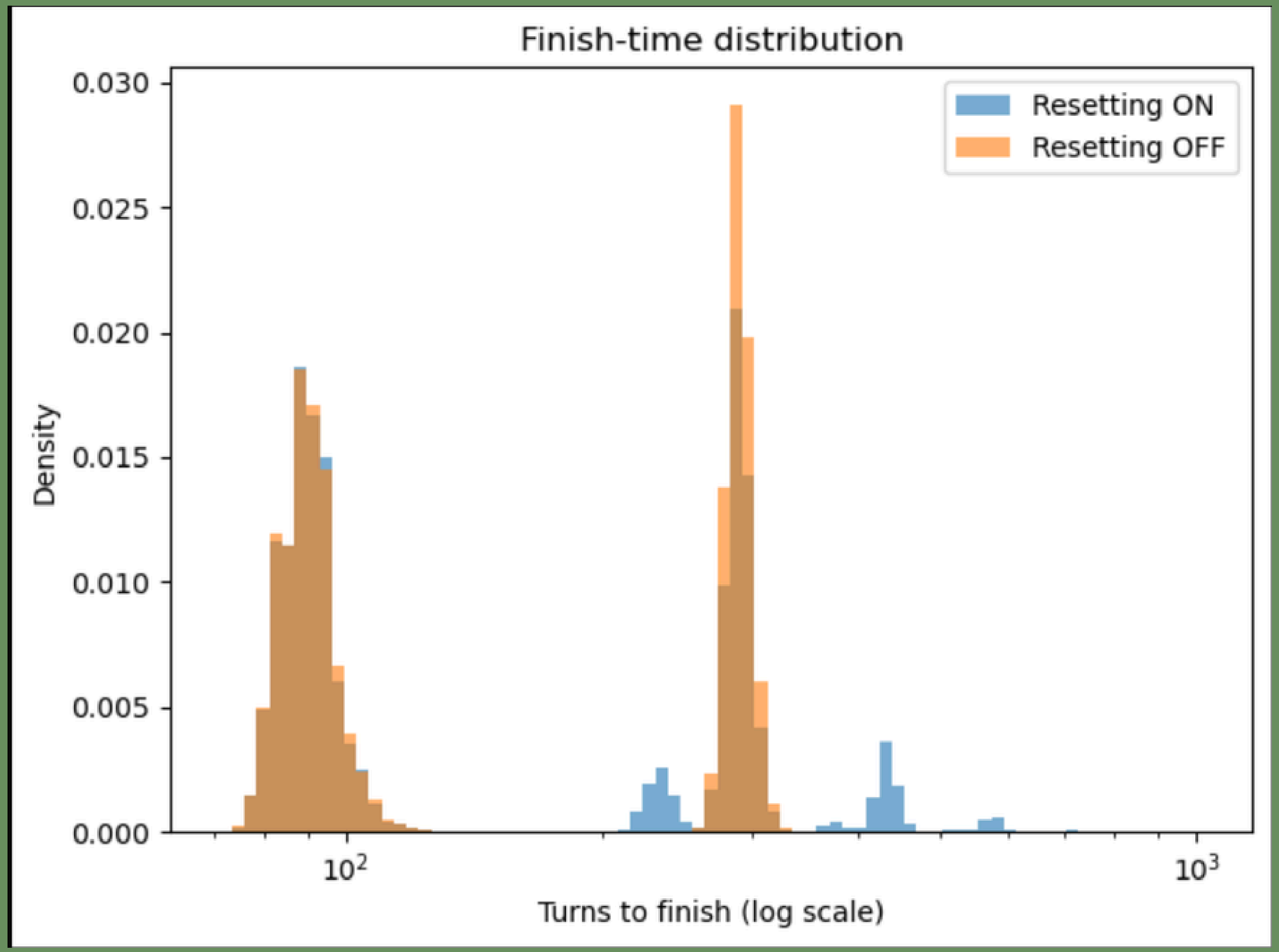
0 ■■■ 40 ■■■ 120 ■■■ 850 ■■ 930 ■■■ 1000



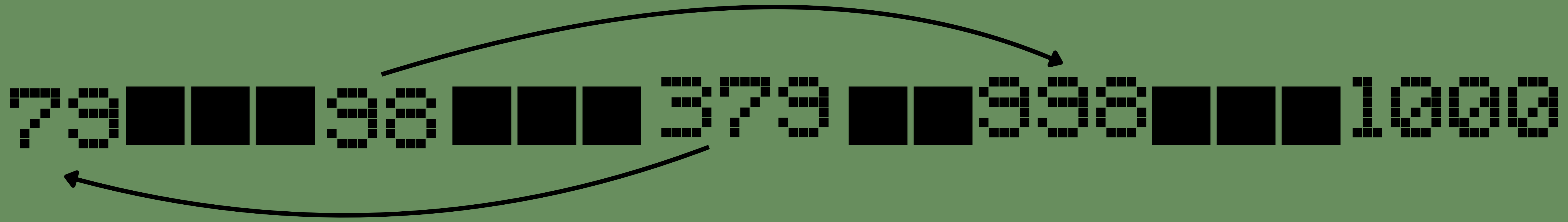


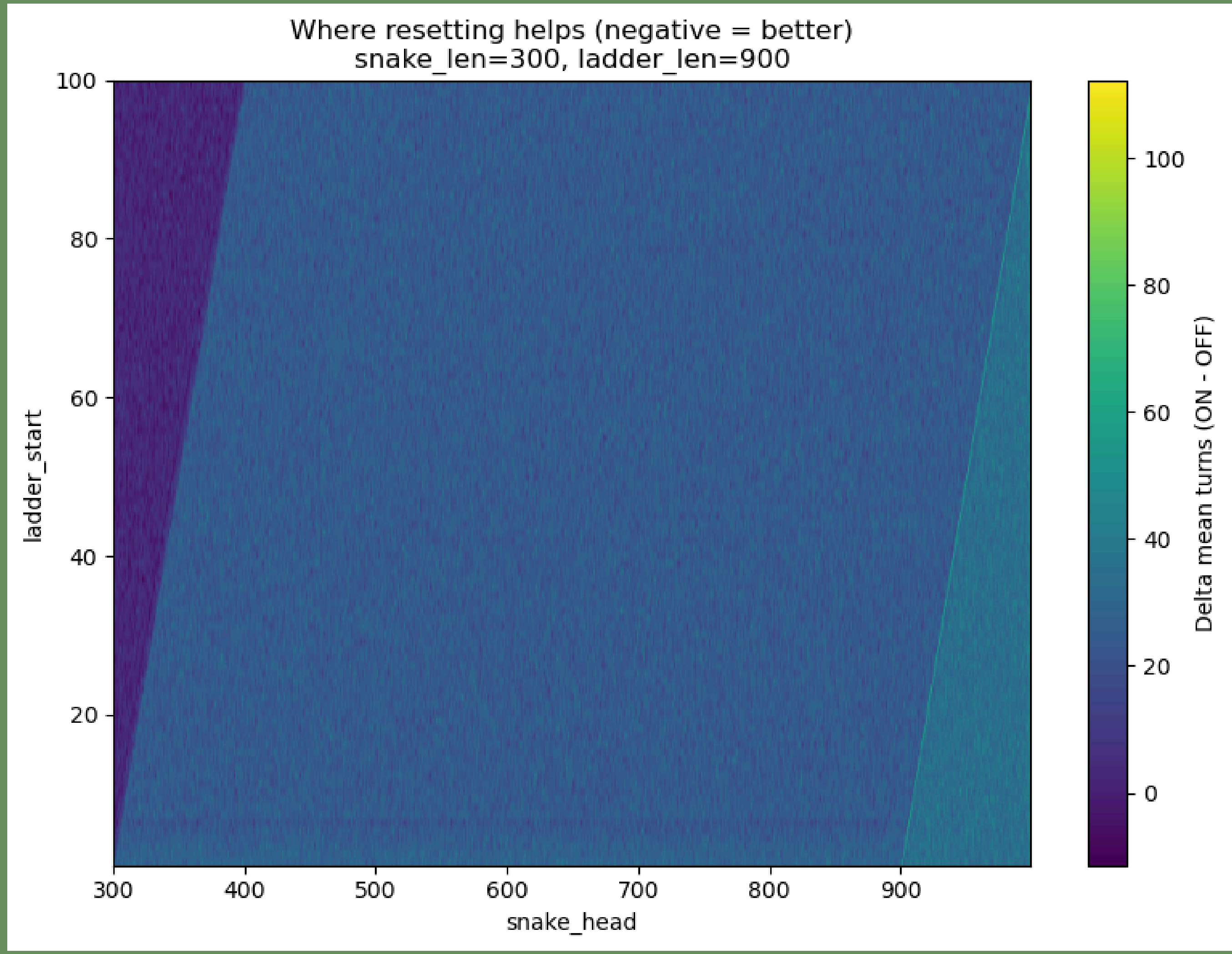
REDUCING SNAKE LENGTH N = 4000

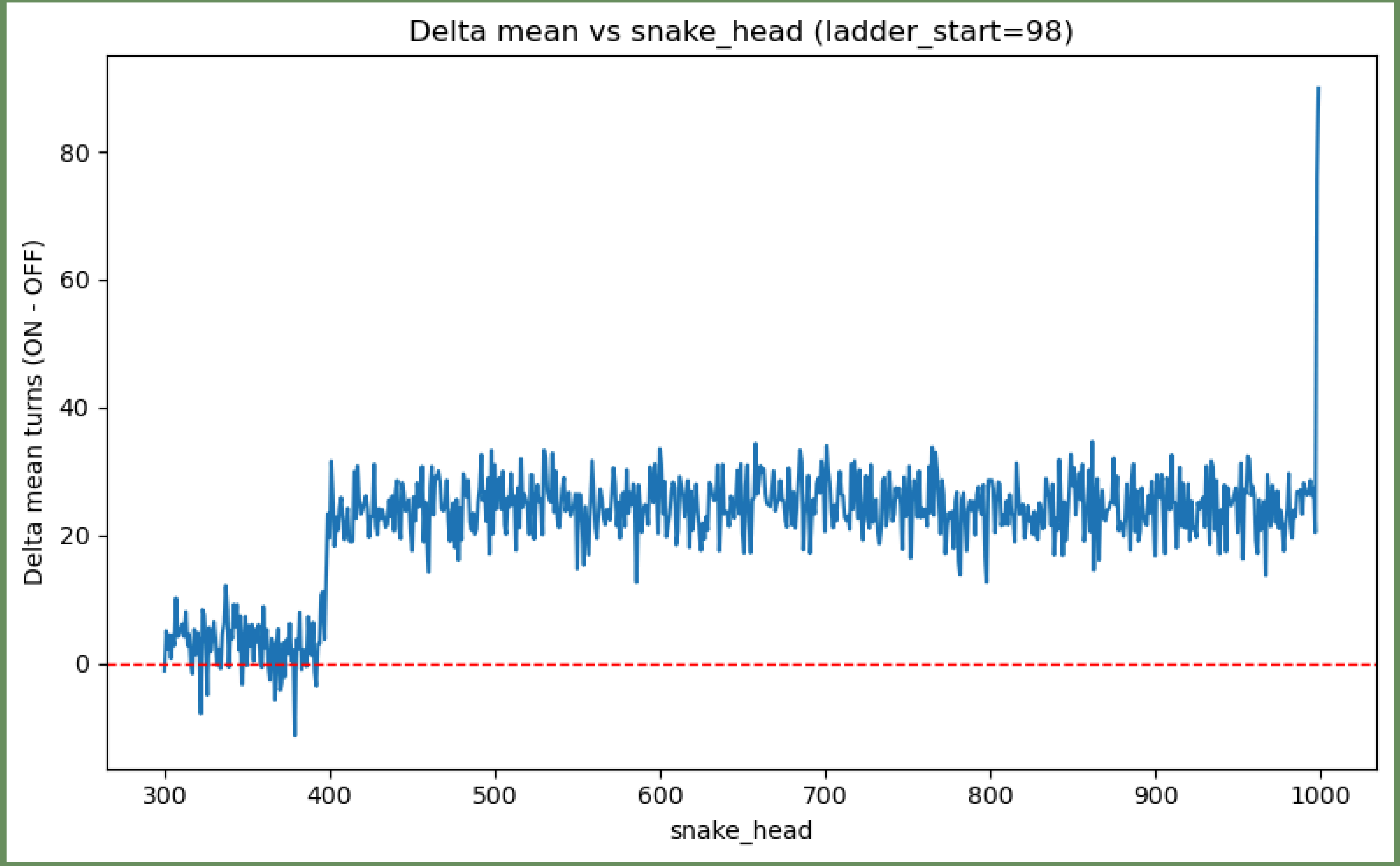




PROMISING RESULTS





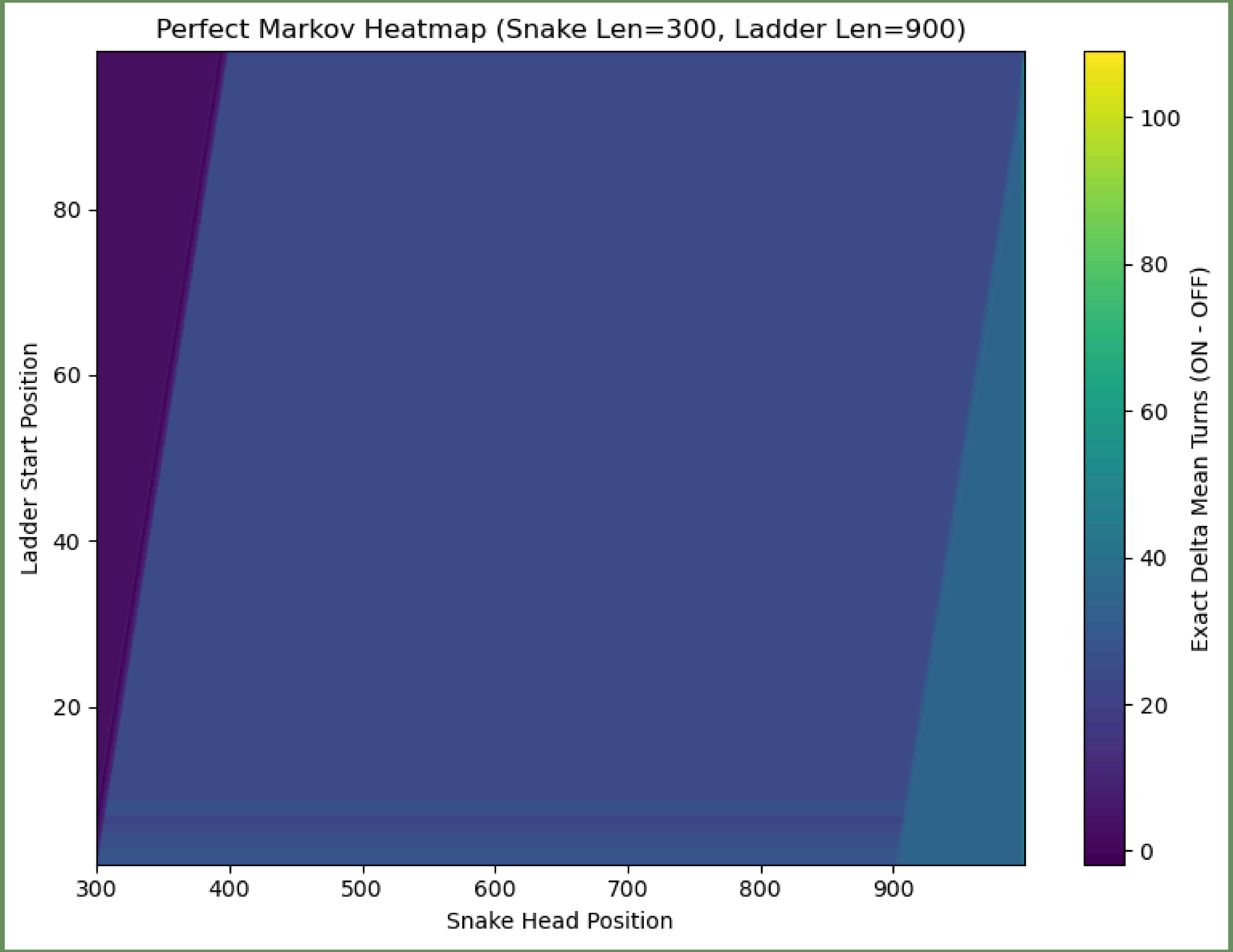


delta (ON - OFF) : -11.393 (negative means resetting helped)

GRAPH USING MARKOVIAN FORMULA

Delta : -1.8050

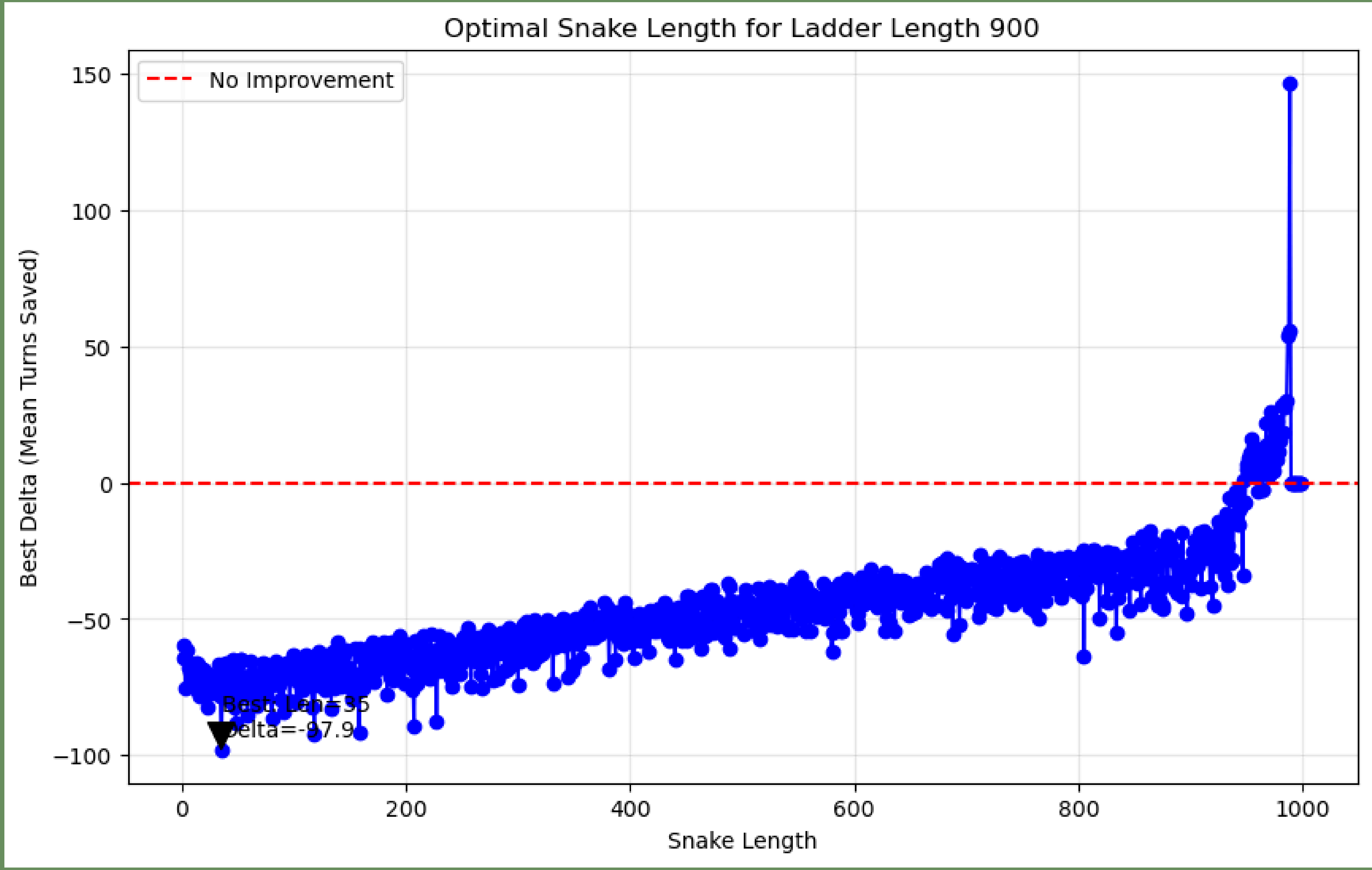
EXACT turns saved



FINDING IDEAL LENGTH

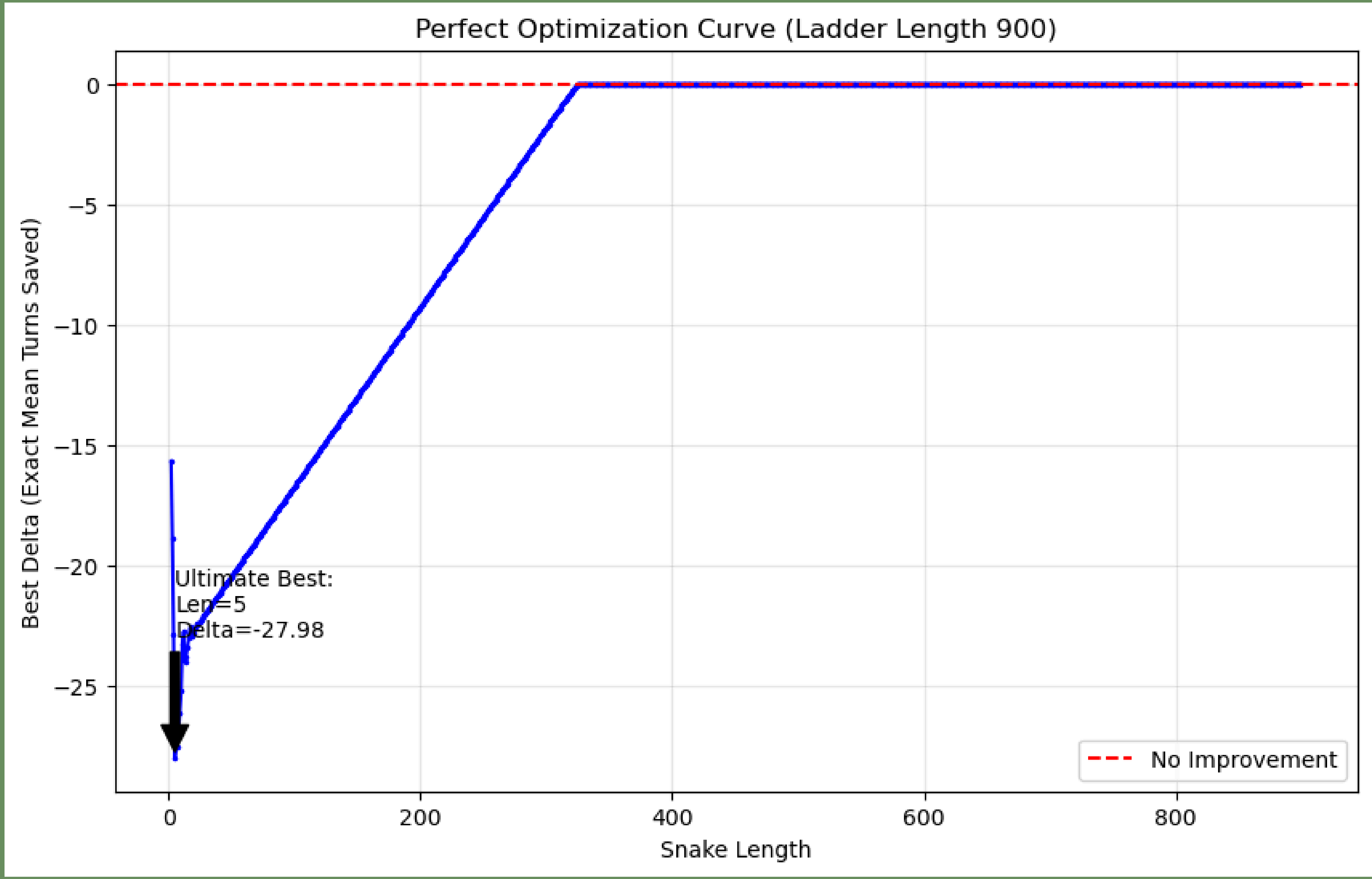
- Now that we saw that snakes can help it reach early, We tried to find the ideal snake position for a given snake and ladder length
- I fixed the ladder length at 900 and snake length at 300 and let it run over the matrix with 2000 trials per setting and 1 ladder and snake step

RUNNING IT OVER SNAKE LENGTH



Now I tried running it over snake length with 3000 trials per configuration and plotted the Least MFPT for each and found the mean was decreasing

RUNNING IT OVER WITH MARKOV



Now I tried doing in with the Markovian Formula we found earlier and we found that the there is a peak for Len < 5 for snakes

Ladder : 5 -> 905

Snake : 6 -> 1

WHY THE PEAK?

- Probability of going from 1 → 2 is 1/6, similarly from 1 → 3 is ⅙ + ⅙*⅙, for 1 → 4 is ⅙ + 2*⅙*⅙ + ⅙*⅙*⅙ and so on.
- The running theory is right now, 1 → 5 is the sweet spot between maximum probability and the mean number of turns required to get there.
- The next step in the project would be to find an ideal cost function that takes into account both probability and number of turns

THANK YOU

FEEL FREE
TO ASK
QUESTIONS